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Strong convergence of the Mann iteration for α-demicontractive mappings

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ABSTRACT

The paper deals with strong convergence properties of the Mann iteration. A new class of demicontractive mappings (called α-*demicontractive*) is introduced for which the strong convergence of the computed sequence is assured. The paper presents also an overview of relevant contributions of the last two decades, concerning strong convergence for Mann-type iteration of demicontractive mappings.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle ., .\rangle$ and induced norm $\|.\|$ and let $T : \mathcal{C} \to \mathcal{C}$ be a nonlinear mapping, where $\mathcal{C}$ is a closed convex subset of $\mathcal{H}$. We will suppose throughout the paper that the set of fixed points of T is nonempty, $\mathrm{Fix}(T) \neq \emptyset$.

Definition 1 (*[1]*)**.** The mapping T is said to be demicontractive (or demicontractive with constant k) if

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T). \tag{1.1}$$

Usually, the constant k is assumed to be in the interval $(0, 1)$, but often this restriction is avoided; for example, in [2] is considered a class of mappings satisfying (1.1) with negative values of k and called *strongly attracting maps*; a map T satisfying (1.1) with $k = 1$ is called *hemicontractive*, and this condition was used in [3,4] to prove the strong convergence of the implicit Mann iteration introduced in [5].

Definition 2 (*[6]*)**.** The mapping T is said to satisfy condition (A) if

$$\langle x - Tx, x - p\rangle \geq \lambda\|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T), \tag{1.2}$$

where λ is a positive real number.

Note that (1.2) is a condition of accretive (or monotone) type.

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It is easy to see that conditions (1.1) and (1.2) are equivalent. Indeed, by a direct computation, we obtain

$$\|x-p\|^2+k\|x-Tx\|^2-\|Tx-p\|^2=2\left(\langle x-Tx,x-p\rangle-\frac{1-k}{2}\|x-Tx\|^2\right),$$

and the equivalence is obvious with $\lambda=(1-k)/2$. Thus the class of demicontractive mappings coincides with the class of mappings satisfying condition (A). If $k\in(0,1)$ then $\lambda\in(0,1/2)$.

Note that the class of demicontractive mappings is one of the most general classes for which some iterative methods were investigated; it includes the well-known class of quasi-nonexpansive mappings and the class of strictly pseudocontractive mappings of Browder–Petryshyn type with nonempty fixed point set.

It is worthwhile to mention that the concept of demicontractivity is especially used in the study of Mann-type iteration. Recall that the Mann iteration is given by the iteration formula

$$x_{n+1}=(1-t_n)x_n+t_nTx_n,\quad x_0\in\mathcal{C}, \tag{1.3}$$

where $\{t_n\}$ is a real sequence in [0, 1] satisfying some appropriate conditions, which is usually called *control sequence*. If $t_n=1/2,\forall n\in\mathbb{N}$, then (1.3) becomes $x_{n+1}=(x_n+Tx_n)/2$, which is the well-known method of Krasnoselski. In what follows, the term Krasnoselski–Mann iteration (or KM iteration) will be used for (1.3) as well.

A more general iteration is the *Ishikawa iteration*, given by

$$\begin{aligned}y_n&=(1-\alpha_n)x_n+\alpha_nTx_n,\\x_{n+1}&=(1-\beta_n)x_n+\beta_nTy_n,\end{aligned}$$

where the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy some appropriate conditions. In particular, when $\alpha_n=0,\forall n\in\mathbb{N}$, the Ishikawa iteration becomes the standard KM iteration.

Demicontractivity alone is not sufficient for the convergence of the sequence generated by the KM iteration, even in finite dimensional spaces; some additional smoothness properties of T are necessary, like continuity or demiclosedness. Recall that a mapping T is said to be demiclosed at zero (or that T satisfies the demiclosedness principle), if for any sequence $\{x_n\}$ such that $x_n\to p$ (weakly) and $Tx_n\to 0$ (strongly), $Tp=0$ holds. These two conditions, together with certain boundedness conditions on the control sequence, assure the convergence in finite dimensional spaces. However, in infinite dimensional spaces these conditions are not sufficient (see Section 2).

The paper is organized as follows. In Section 2, we provide a short overview of relevant research contributions concerning weak and strong convergence of demicontractive mappings. Section 3 deals with a special condition for getting strong convergence. In the remainder of the paper (Section 4), we introduce a new class of mappings, called *α-demicontractive*, for which the KM iteration has strong convergence in the frame of a real Hilbert space. Some properties of the new class are also studied.

2. A short overview of the weak and strong convergence of the KM iteration for demicontractive mappings

In finite dimensional spaces, the convergence result of KM iteration for the demicontractive case was given by Măruşter [7], and applied to the study of the so-called *relaxation algorithm* for the solution of a particular convex feasibility problem [8,9]. The general convex feasibility problem is usually solved by the projection type algorithm which is a particular KM iteration; note that the convex feasibility problem has relevant practical applications (see [2]) which motivate the interest for the KM iteration.

The main result of [7] is

Theorem 1. *Let $T:\mathbb{R}^m\to\mathbb{R}^m$ be a nonlinear mapping, where $\mathbb{R}^m$ is the Euclidean m-dimensional space. Suppose the following conditions are satisfied:*

(i) *I–T is demiclosed at* 0*;*
(ii) *T is demicontractive with constant k, or, equivalent, T satisfies the condition* (A) *with $\lambda=(1-k)/2$;*
(iii) $0<a\le t_n\le b<2\lambda=1-k$.

Then the sequence $\{x_n\}$ given by the KM *iteration converges to a point of* Fix(T) *for any starting point x_0.*

Remark 1. In [7] this theorem is presented slightly differently, more precisely: $T=I-U$, the iteration is $x_{n+1}=x_k-tUx_n$ and the set of fixed points of T coincides with the solutions S of $Ux=0$. The condition (ii) appears in the following form:

$$\inf_{\substack{x\in\mathcal{R}^n\setminus S\\ \xi\in S}}\frac{\langle x-\xi,Ux\rangle}{\|Ux\|^2}=\eta>0.$$

It is easy to see that this condition is equivalent to (ii).

2.1. Weak convergence

In infinite dimensional spaces, the two conditions (demiclosedness and demicontractivity) are not sufficient for strong convergence; Genel and Lindenstraus [10] provided an example of a contraction defined on a bounded closed convex subset of a Hilbert space for which the Krasnoselski iteration does not converge. However, the two conditions ensure weak convergence. In real Hilbert spaces this result was given in [1,6]:

Theorem 2. *Let $T : C \to C$ be a nonlinear mapping, where C is a closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose the following conditions are satisfied:*

(i) *I–T is demiclosed at* 0*;*
(ii) *T is demicontractive with constant k, or, equivalent, T satisfies the condition* (A) *with $\lambda = (1 - k)/2$;*
(iii) $0 < a \le t_n \le b < 2\lambda = 1 - k$.

Then the sequence $\{x_n\}$ given by the KM iteration converges weakly to a point of $\mathrm{Fix}(T)$ *for any starting point x_0.*

Note that this result is a direct generalization of Theorem 1 from finite dimensional spaces to real Hilbert spaces.

The class of demicontractive mappings and the KM iteration were studied by several authors (see the survey paper [11]) and are still being studied in some recent papers (see, for example, [12,13]). The generalized concept of demicontractivity is also studied. For example, in [14], the following generalization is considered. Let E be a real q-uniformly smooth Banach space ($q > 1$), which admits a weakly sequentially continuous duality map J_q.

Definition 3. A mapping $T : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is a closed convex subset of E, is said to satisfy the condition (A*) if $\mathrm{Fix}(T) \ne \emptyset$ and for $j(x - p) \in J_q(x - p)$, there exists $\lambda > 0$ such that

$$\langle x - Tx, j(x - p)\rangle \ge \lambda \|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T).$$

Theorem 3 (*[14]*)**.** *Let $\mathcal{C}$ be a closed convex subset of E and let $T : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition* (A*) *and such that I–T is demiclosed at zero. Let α_n and β_n be two sequences satisfying some appropriate conditions (see Lemma* 1 *from* [14]*). Then the sequence $\{x_n\}$ given by Ishikawa iteration converges weakly to a fixed point of T.*

2.2. Strong convergence

Considering accomplished the two conditions, T is demicontractive and demiclosed at zero, in order to obtain strong convergence, some additional conditions or some modifications of the standard KM iteration are necessary.

Additional conditions were provided in some old and recent papers. For example, in [1] the following additional condition is required: I–T maps closed bounded subsets of $\mathcal{C}$ into closed subsets of $\mathcal{C}$ (in particular, this is satisfied if T is demicompact). Mann and Ishikawa iteration processes with errors for demicontractive mappings were shown to converge strongly if $\mathcal{C}$ is a compact convex subset and the error terms satisfy some boundedness conditions [15]. Recently, Boonchari and Saejung [12] require that T must be a Lipschitzian and demicompact mapping.

In order to obtain strong convergence, *modifications* of the Mann iteration were performed. One of the most famous modification is the (CQ) algorithm given by Nakajo and Takahashi [16]. In its essence, this algorithm consists in the projection of the initial iteration onto the intersection of suitably constructed sets C_n and Q_n, which depend on the current iteration x_n at every step (that is the computation of $P_{C_n \cap Q_n} x_0$). In the case of demicontractive mappings class, the strong convergence of the algorithms of this type was proved, for example by Quin et al. [17] (the authors use the term *quasi-strict pseudocontractivity* for the old concept of demicontractivity).

Another modification which ensures the strong convergence is the implicit KM iteration, developed in [4,3]. The iteration formula (1.3) is modified by replacing the term Tx_n with Tx_{n+1} such that the iteration becomes implicit.

A particular modification was proposed by Schu [18] for *asymptotically nonexpansive* mappings. This concept was extended to that of *asymptotical demicontractivity* [19] as follows: let $\{a_n\}$ be a sequence of real numbers such that $a_n \ge 1$ and $\lim a_n = 1$ and let k be some constant in $[0, 1)$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is asymptotically demicontractive with sequence $\{a_n\}$ and constant k if

$$\|T^n x - p\|^2 \le a_n^2 \|x - p\|^2 + k\|x - T^n x\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T).$$

It is shown that the sequence $\{x_n\}$ given by the iteration process considered in [19] converges strongly to some fixed point of T, provided that T is completely continuous, uniformly L-Lipschitzian and asymptotically demicontractive.

Osilike [20] proved that the condition of asymptotical demicontractivity is equivalent to

$$\langle x - T^n x, x - p\rangle \ge \frac{1}{2}(1 - k)\|x - T^n x\|^2 - \frac{1}{2}(a_n - 1)\|x - p\|^2.$$

Note that in particular, if $a_n = 1$, $\forall n$, then this condition is similar to condition (1.2).

Remark 2. Both (CQ) algorithm, the implicit KM iteration and the asymptotically nonexpansive (or demicontractive) variants, require a significant computational effort. For example, in the (CQ) algorithm, the value of $P_{C_n \cap Q_n} x_0$ cannot be obtained easily, in general; in the implicit KM iteration, the computation of the next iteration x_{n+1} involves solving a nonlinear equation at every step of the iteration, task which may pose the same difficulty level as the initial problem; finally, the asymptotically nonexpansive (or demicontractive) variants require the computation of $T^n x_n$ at every step, which may be computationally expensive when the iterations are large. Very recently, He et al. [21] propose an interesting method of realization of the (CQ) algorithm; the basic idea is to get an explicit expression of $P_{C_n \cap Q_n} x_0$, so that laborious computation of the sets C_n, Q_n, $C_n \cap Q_n$ and $P_{C_n \cap Q_n} x_0$ are avoided.

In a recent paper [13] Maingé proposed a very simple modification (at least from the computation point of view) which ensures the strong convergence. He called it *regularized* KM *algorithm*, and it has the following very simple iterative formula

$$\begin{cases} v_n = (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)v_n + \beta_n T v_n, \end{cases}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy some appropriate conditions. These results were developed very recently in [22].

Remark 3. The regularized KM iteration is almost a common KM iteration. As $\alpha_n \to 0$, the asymptotic behavior of the sequence generated by this iteration is like the sequence generated by the KM iteration. It is therefore surprising that this sequence converges strongly to a fixed point in the common conditions (T demicontractive and I–T demiclosed at 0). On the other hand, the sequence $\{\alpha_n\}$ must satisfy the following two conditions: $\lim \alpha_n = 0$ and $\sum \alpha_n \to \infty$, so $\{\alpha_n\}$ is a slow decreasing sequence which would explain the good behavior of the regularized KM iteration.

3. Strong convergence and variational inequalities

There is an interesting connection between the strong convergence of the Mann iteration and the existence of a nonzero solution of a certain variational inequality. This connection was observed by Măruşter [6] and reconsidered by some authors. In [6] the existence of a nonzero solution $h \in \mathcal{H}$, $h \neq 0$, of the variational inequality

$$\langle x - Tx, h\rangle \leq 0, \quad \forall x \in \mathcal{C} \tag{3.1}$$

is required as an additional condition for strong convergence. More precisely, the strong convergence theorem in [6] (Theorem 2) is

Theorem 4. *Suppose T satisfies the conditions of Theorem 2. If in addition there exists a nonzero solution of (3.1), $h \in \mathcal{H}$, $h \neq 0$, then for suitable x_0 the sequence $\{x_n\}$ converges strongly to an element of* Fix(T).

It is obvious that the existence of a nonzero solution of this variational inequality occurs only in very particular cases; an example for linear equations is given in [6].

Remark 4. The variational inequality $\langle x - Tx, h\rangle \leq 0$, $\forall x \in \mathcal{C}$ is not in a standard form. This variational inequality can be brought to the standard form in the following way. The considered variational inequality is equivalent to

$$\langle Tx, x\rangle - \|x\|^2 + \langle x - Tx, x - h\rangle \geq 0, \quad \forall x \in \mathcal{C}.$$

Supposing that T satisfies the positivity type condition $\langle Tx, x\rangle \geq \|x\|^2$, it is sufficient to find a solution of the variational inequality $\langle x - Tx, x - h\rangle \geq 0$. If I–T is accretive (i.e. $\langle (I-T)u - (I-T)v, u - v\rangle \geq 0$, $\forall u, v \in \mathcal{C}$), then it is sufficient to find a solution for $\langle h - Th, x - h\rangle \geq 0$. But this last inequality is satisfied if $h \in$ Fix(T), that is if Fix$(T) \neq \emptyset$.

Note that the two conditions (the positivity of T and the accretivity of I–T) are very strong; so, it should be interesting to find some weaker conditions.

Theorem 4 was extended in some papers [23,14,24,25].

For example Chidume [23] presents the following generalization. Let E be a real Banach space with the property $(U, \alpha, m+1, m)$ (see [23]) and which admits a weakly continuous duality map. Suppose that the mapping $T : K \to K$, where K is a closed convex subset of E, satisfies the condition (A∗) and that I–T is demiclosed at 0. Suppose further that there exists $h \in K$, $h \neq 0$ such that $\langle u, x - Tx\rangle \leq 0$, $u \in J(h)$, for all $x \in K$. Then the sequence generated by Mann iteration, where the control sequence $\{t_n\}$ satisfies $0 < a \leq t_n \leq b < (\frac{\alpha}{2^m - 1})^{-1}\lambda^m(m+1)$ and for suitable $x_0 \in K$, converges strongly to a fixed point of T.

For the Ishikawa iteration a similar result was given in [14]. Let E be a real q-uniformly smooth Banach space ($q > 1$), which admits a weakly continuous duality map and let K be a closed convex subset of $\mathcal{H}$. Suppose that $T : K \to K$ satisfies the condition (A*) and that I–T is demiclosed at 0. Suppose further that there exists $h \in K$, $h \neq 0$ such that $\langle x - Tx, j(h)\rangle \leq 0$, for all $x \in K$. Then for a suitable $x_0 \in K$ the sequence $\{x_n\}$ generated by Ishikawa iteration converges strongly to a fixed point of T. The real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are supposed to satisfy some appropriate conditions (see [14]).

It would therefore be interesting to study more closely the existence of a nonzero solution of the variational inequality (3.1).

4. α-demicontractivity

We first introduce a new concept of demicontractivity, called α-*demicontractivity*:

Definition 4. The mapping T is said to be α-demicontractive if

$$\langle x - Tx, x - \alpha p\rangle \geq \lambda \|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T), \tag{4.1}$$

for some $\alpha \geq 1$.

Remark 5. It is easy to see that (4.1) is equivalent with

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + k\|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T).$$

Theorem 5. *Suppose T satisfies the conditions of Theorem 2. Suppose also that T is α-demicontractive for some $\alpha > 1$. Then for suitable x_0, the sequence $\{x_n\}$ converges strongly to a point in* $\mathrm{Fix}(T)$.

Proof. It follows from Theorem 2 that $\{x_n\}$ converges weakly to some $p \in \mathrm{Fix}(T)$.

First, by mathematical induction, we will show that $\langle x_n - p, p\rangle \geq \frac{1}{2(\alpha-1)}\|x_n - p\|^2,\ \forall n \in \mathbb{N}$.

Let x_0 be such that $\langle x_0 - p, p\rangle \geq \frac{1}{2(\alpha-1)}\|x_0 - p\|^2$, which is the base of induction. The existence of an x_0 which satisfies this inequality can be proved in different ways. For example, let β be a real number such that $1 < \beta \leq 2\alpha - 1$ and take $x_0 = \beta p$; then this x_0 is a suitable starting point (see also Remark 6).

Assuming that $\langle x_n - p, p\rangle \geq \frac{1}{2(\alpha-1)}\|x_n - p\|^2$ (inductive hypothesis), using (4.1) and the conditions of Theorem 2, we will show that $\langle x_{n+1} - p, p\rangle \geq \frac{1}{2(\alpha-1)}\|x_{n+1} - p\|^2$, which is the inductive step.

We obtain successively:

$$\langle x_n - Tx_n, x_n - \alpha p\rangle \geq \lambda\|x_n - Tx_n\|^2 \geq t_n/2\|x_n - Tx_n\|^2;$$
$$\langle x_n - Tx_n, x_n - p - (\alpha - 1)p\rangle \geq t_n/2\|x_n - Tx_n\|^2;$$
$$-t_n\langle x_n - Tx_n, p\rangle \geq -\frac{t_n}{\alpha - 1}\langle x_n - Tx_n, x_n - p\rangle + \frac{t_n^2}{2(\alpha - 1)}\|x_n - Tx_n\|^2.$$

Adding the inductive hypothesis, it follows that

$$\langle x_n - p, p\rangle - t_n\langle x_n - Tx_n, p\rangle \geq \frac{1}{2(\alpha - 1)}\|x_n - p\|^2 - \frac{t_n}{\alpha - 1}\langle x_n - Tx_n, x_n - p\rangle + \frac{t_n^2}{2(\alpha - 1)}\|x_n - Tx_n\|^2.$$

That is

$$\langle x_n - t_n(x_n - Tx_n) - p, p\rangle \geq \frac{1}{2(\alpha - 1)}\|x_n - t_n(x_n - Tx_n) - p\|^2.$$

Therefore

$$\langle x_{n+1} - p, p\rangle \geq \frac{1}{2(\alpha - 1)}\|x_{n+1} - p\|^2,$$

which is the required inequality.

Now, using Theorem 2, we have $\|x_n - p\| \to 0$. □

Remark 6. The existence of suitable x_0 can also be proved as follows. Suppose that O (the origin of space) does not belong to the set $\mathrm{Fix}(T)$, and let y be the projection of O onto the set of fixed points (if T is demicontractive, the set $\mathrm{Fix}(T)$ is closed and convex, so the projection exists). Then $\langle y, p - y\rangle \geq 0$. Now, take $x_0 = p + 2(\alpha - 1)y$, or $y = \frac{1}{2(\alpha-1)}(x_0 - p)$. Putting this value of y into the above mentioned inequality, we get the required base of induction.

It is worthwhile to know further the characteristics of an α-demicontractive mapping. Namely, how large the class of α-demicontractive mappings is, and also its connection with the demicontractive class.

Therefore, we will state below some facts concerning α-demicontractive mappings.

Fact 1. *If T is α-demicontractive then αp is a fixed point of T;* this is obvious.

Fact 2. *If T is demicontractive and the variational inequality (3.1) has the fixed point p as a nonzero solution, then T is α-demicontractive.*

Proof. From demicontractivity and from $\langle x - Tx, p\rangle \leq 0,\ \forall x \in \mathcal{C}$, we have

$$\langle x - Tx, x - p\rangle - \lambda\|x - Tx\|^2 \geq (\alpha - 1)\langle x - Tx, p\rangle, \quad \forall x \in \mathcal{C},$$

for any $\alpha > 1$, which is (4.1). □

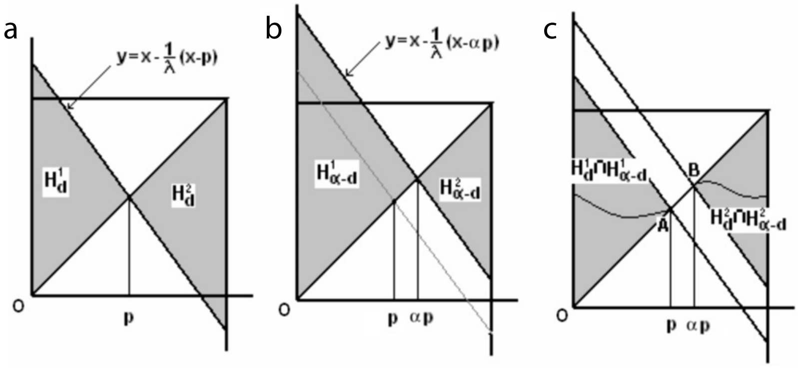

Fig. 1. The case of a demicontractive and α-demicontractive for a real function.

Fact 3. *If T is α-demicontractive, then $\langle y - Ty, p\rangle \le 0,\ \forall y = (1-t)p + t\alpha p$.*

Proof. Because $y - \alpha p = (1-\alpha)(1-t)p$, the α-demicontractivity property yields

$$(1-\alpha)(1-t)\langle y - Ty, p\rangle \ge \|x - Tx\|^2,$$

which implies $\langle y - Ty, p\rangle \le 0$. □

Fact 4. *If T is a real function, $T = f : [0, 2p] \to [0, 2p]$, where p is a fixed point of f, the demicontractivity and α-demicontractivity, $0 < \lambda < 1/2$ and $1 < \alpha < 2$, imply $f(x) = x,\ \forall x \in [p, \alpha p]$ (that is any point belonging to the segment $[p, \alpha p]$ is a fixed point of f, $[p, \alpha p] \subset \mathrm{Fix}(f)$).*

Proof. It is elementary to see that f is demicontractive with constant k (or that f satisfies condition (A) with $\lambda = (1-k)/2$) if and only if

$$\begin{cases} x \le f(x) \le x - \dfrac{1}{\lambda}(x-p), & \text{if } x \le p,\\ x - \dfrac{1}{\lambda}(x-p) \le f(x) \le x, & \text{if } x > p.\end{cases}$$

This means that the graph of f must belong to the shaded region $H_d^1 \cup H_d^2$, Fig. 1(a).

In a similar way, f is α-demicontractive if and only if

$$\begin{cases} x \le f(x) \le x - \dfrac{1}{\lambda}(x-\alpha p), & \text{if } x \le \alpha p,\\ x - \dfrac{1}{\lambda}(x-\alpha p) \le f(x) \le x, & \text{if } x > \alpha p.\end{cases}$$

This means that the graph of f must belong to the shaded region $H_{\alpha-d}^1 \cup H_{\alpha-d}^2$, Fig. 1(b).

Therefore, f is both demicontractive and α-demicontractive if and only if the graph of f belongs to the intersection $(H_d^1 \cup H_d^2) \cap (H_{\alpha-d}^1 \cup H_{\alpha-d}^2)$, that is, to the shaded region in Fig. 1(c), and for $x \in [p, \alpha p]$ the graph must coincide with the main bisector, that is, $f(x) = x$. □

Remark 7. The above analysis allows us to reveal some properties of the α-demicontractive class of mappings. For example, there exist α-demicontractive mappings which are not demicontractive. The real function $f : [0, 2p] \to [0, 2p]$, p-fixed point, defined by

$$f(x) = \begin{cases} 2p, & \text{if } 0 \le x \le \dfrac{\alpha - 2\lambda}{1-\lambda}p,\\ x - \dfrac{1}{\lambda}(x - \alpha p), & \text{if } \dfrac{\alpha - 2\lambda}{1-\lambda}p \le x \le \alpha p,\\ \alpha p, & \text{if } \alpha p \le x \le 2p\end{cases}$$

is such a function.

In the case of a real demicontractive function, $f : [0, 2p] \to [0, 2p]$, p-fixed point of f, there is a certain connection between α-demicontractivity and the condition of the existence of a nonzero solution of (3.1). Observe that, in this case, the existence of a nonzero solution holds if and only if the function $I - f$ preserves the sign on $[0, 2p]$; the graph of f must be either above or below the main bisector. Thus, if $f(x) \ge x,\ \forall x \in [0, 2p]$ then f is α-demicontractive for any $\alpha \in (1, 2]$. If $f(x) \le x,\ \forall x \in [0, 2p]$, then the α-demicontractivity conditions are satisfied only on $[0, p]$.

Thus, in this case (f is a real demicontractive function), the α-demicontractivity is, to some extent, less restrictive than the condition of the existence of a nonzero solution.

Remark 8. In Theorem 5, T is required to be both demicontractive and α-demicontractive for some $\alpha > 1$. In the case of a real function, $[p, \alpha p] \subset \mathrm{Fix}(f)$ (Fact 4). Therefore it would be interesting to know if this property holds in a more general setting of a real Hilbert space.

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